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More polytopes meeting the conjectured Hirsch bound

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Abstract

In 1957, W.M. Hirsch conjectured that every (convex) d-polytope with n facets has edge-diameter at most $n-d$. Recently, Holt and Klee constructed polytopes which meet this bound for a number of pairs (d,n) with $d\leqslant 13$ and for all pairs (d,n) with $d\geqslant 14$. These constructions involve a judicious use of truncation, wedging, and blending on polytopes which already meet the Hirsch bound. In this paper we extend these techniques to construct polytopes of edge-diameter $n-d$ for all (d,n) with $d\geqslant 8$. The improvement from $d=14$ to $d=8$ follows from identifying circumstances in which the results for wedging when $n>2d$ can be extended to the cases $n\leqslant 2d$.

1. Introduction

For two vertices x and y of a polytope P, the *distance* $\delta_P(x,y)$ is defined as the smallest number of edges of P that can be used to form a path from x to y. The *edge-diameter* $\delta(P)$ of P is the maximum over all pairs (x,y) of P's vertices. An undirected edge $[u,v]$ in a polytope P is said to be *slow toward* a vertex x of P iff $\delta_P(u,x)=\delta_P(v,x)$; otherwise, $[u,v]$ is *fast toward* x. Two vertices of a polytope will be called *estranged* iff they do not share a facet. $\Delta(d,n)$ denotes the maximum edge-diameter among all (d,n)-polytopes, where (d,n)-*polytope* means a simple d-dimensional polytope with precisely n facets; a polytope is *simple* iff each vertex is incident to precisely d facets.

As reported by Dantzig [3,4], W.M. Hirsch conjectured in 1957 that $\Delta(d,n)\leqslant n-d$ for all $n>d\geqslant 2$. His conjecture originally addressed both unbounded objects (polyhedra) and bounded ones (polytopes). The Hirsch conjecture holds [9] for $d=3$ and all n, even in the unbounded version. In the bounded case it holds whenever $n-d\leqslant 5$, but the unbounded version fails for $(d,n)=(4,8)$; both of these results appear in [11]. We

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know two other specific values [5]: $\Delta(4,10)=5$ and $\Delta(5,11)=6$. The bounded Hirsch conjecture is still open for $(4,n>10)$, for $(5,n>11)$ and for all (d,n) with $d\geqslant 6$ and $n-d>5$. The bounded Hirsch conjecture is equivalent to the *d-step conjecture*, that $\Delta(d,2d)=d$ for all d. For a background on the Hirsch conjecture and references to the extensive literature see [10].

We say that (d,n) is sharp for the Hirsch conjecture or *H-sharp* if $\Delta(d,n)\geqslant n-d$. It is well known [9] that all (d,n) with $d<n\leqslant 2d$ are H-sharp, that $\Delta(2,n)=\lfloor\frac{1}{2}n\rfloor$, $\Delta(3,n)=\lfloor\frac{2}{3}n\rfloor-1$ and thus for $d\leqslant 3$, $n\leqslant 2d$ is also necessary for H-sharpness. Hence we focus on pairs for which $d\geqslant 4$ and $n>2d$.

Of the 1142 combinatorial types of (4,9)-polytopes catalogued by Altshuler, Bokowski and Steinberg [2], only one (first constructed in [11]) has diameter 5, demonstrating that (4,9) is H-sharp. Here, as in [7,8], that one is denoted by Q_4.

In [8] the authors initiated a tabulation of H-sharp pairs (d,n); these were obtained from Q_4 under the operations of truncation, wedging and fast-slow blending, which we will describe in Section 2. It was shown that various pairs (d,n) with $d\leqslant 13$ are H-sharp, but the most important result was the fact that when $d\geqslant 14$, the pair (d,n) is H-sharp for all $n>d$. We will show in Section 4 of this note that (d,n) is H-sharp for all $n>d\geqslant 8$.

We say that two subsets X and Y of vertices in an H-sharp (d,n)-polytope P form an *H-pair* iff $\delta_P(x,y)\geqslant n-d$ for all (x,y) in $X\times Y$. *X holds a k-face* iff there is a k-face of P whose vertices all belong to X. We denote by $(d,n\colon h,k)$ the set of all triples (P,X,Y) in which P is an H-sharp (d,n)-polytope with an H-pair (X,Y) such that X holds an h-face and Y holds a k-face. The collection of all nonempty quadruples $(d,n\colon h,k)$ is denoted by $\mathscr{T}$. For instance, Q_4 demonstrates that $(4,9\colon 0,0)$ is in $\mathscr{T}$. As in [8], the set of H-sharp (d,n) is denoted by $\mathscr{S}$.

A pair of vertices is said to be *k-distant* iff the distance between them is exactly k. In [12], attention was brought to the number of paths of length $n-d$ which join $(n-d)$-distant vertices of an H-sharp (d,n)-polytope; the minimum for this number of paths is denoted by $\#^{n-d}(d,n)$, so for $(d,2d)$-polytopes this minimum for this number is denoted by $\#^d(d,2d)$. The d-step conjecture only asserts $\#^d(d,2d)>0$; the current [7] upper bound for this minimum is $\#^d(d,2d)\leqslant\frac{3}{8}\cdot 2^d$. Examination of the H-sharp $(8,n)$-polytopes constructed here yields an upper bound for $\#^d(d,2d)$ of $\frac{1}{2}\cdot 24^k\cdot 2^j$ for $d=5k+j$, which asymptotically yields $\#^d(d,2d)\in\mathscr{O}(1.88817502259^d)$.

2. Three basic operations

To produce H-sharp polytopes from other H-sharp polytopes we use the operations of *wedging*, *truncation* and *fast-slow blending* presented in [7]. Here we will give only a short description. The constructions can be found in [11,7,8,1]. For completeness in this exposition, we include results from [7,8], but for brevity we do not duplicate their proofs.

2.1. Wedging

Suppose that P is a (d,n)-polytope and F is any facet of P. The *wedge over P with foot F* is a $(d+1,n+1)$-polytope $\omega_F P$ that is formed by intersecting the "cylinder" $C = P \times [0,\infty[$ with a hyperplane H in the space spanned by C such that $H \cap P = F$ and H intersects the interior of C. From the two resulting polyhedra we choose $\omega_F P$ to be the bounded one, which contains P. The facets $P \times \{0\}$ and $H \cap C$ will be called the *base B* and the *top T* of the wedge, respectively. Both the base and top are combinatorially equivalent to P. Often the foot F will be clearly indicated by context or restricted to some subset of facets, in which cases we simply write ωP for $\omega_F P$. For any set of vertices X of P, we denote by ωX the union of its images in the base and top of ωP.

The wedge of an H-sharp (d,n)-polytope over any facet produces an H-sharp $(d+1,n+1)$-polytope. However, the following lemma specifies conditions under which a wedge produces from a triple in $(d,n\colon h,k)$ a triple in $(d+1,n+1\colon h+1,k+1)$.

Lemma 2.1. *Let $(P,X,Y) \in (d,n\colon h,k)$. If there exists a pair of vertices (x,y) in $X \times Y$, with x incident to the indicated h-face and y to the indicated k-face, such that there is a facet F of P incident to neither x nor y, then*

$$(\omega_F P, \omega_F X, \omega_F Y) \in (d+1, n+1\colon h+1, k+1).$$

Consequently we have the following implications for membership in S and T.

(i) *If $n > 2d$, then $(d,n) \in \mathscr{S} \Rightarrow (d+1,n+1\colon 1,1) \in \mathscr{T}$;*

(ii) *Given a polytope P with x, y and F as described above, then $(d,n\colon h,k) \in \mathscr{T} \Rightarrow (d+1,n+1\colon h+1,k+1) \in \mathscr{T}$.*

This is Corollary 3.3 from [8].

If $(P,x,y) \in (d,n\colon 0,0)$ with $n > 2d$, then there are at least $n-2d$ facets incident to neither x nor y. In the following lemma we will show, that if $(P,X,Y) \in (d,2d\colon h,k)$ with $h+k > 0$, such a facet F exists for some vertices $(x,y) \in X \times Y$ too.

Lemma 2.2. *If $(d,2d\colon h,k) \in \mathscr{T}$ and $h+k > 0$, then $(d+j,2d+j\colon h+j,k+j) \in \mathscr{T}$ for all $j \geqslant 0$.*

Proof. W.l.o.g. we suppose $h > 0$. Let $(P,X,Y) \in (d,n\colon h,k)$ and let y be a vertex in the k-face held by Y. P is simple, so like every other vertex, y is incident to exactly d facets. Since P has $2d$ facets, there is at most one vertex in P estranged from y. As $h > 0$, there are at least two vertices incident to the h-face held by X. Let x be a vertex of this h-face which is *not* estranged from y. Then there are at most $2d-1$ facets incident to x or y, and therefore there is a facet F of P which is incident to neither x nor y. Hence Lemma 2.1 applies and $(\omega_F P, \omega X, \omega Y) \in (d+1, 2d+1\colon h+1, k+1)$.

In $\omega_F P$ the vertices $x_b := x \times 0$ and $y_b := y \times 0$ are incident to the $(h+1)$-face held by ωX and to the $(k+1)$-face held by ωY, respectively. These vertices, x_b and y_b, are incident to the base B, yet neither is incident to the $(d-1)$-face F in $\omega_F P$, and thus neither to the top T. So successive wedging over the top of the previous wedge produces $(\omega_T^{j-1}\omega_F P, \omega^j X, \omega^j Y) \in (d+j, 2d+j\colon h+j, k+j)$, $j \geqslant 0$. $\square$

2.2. Truncation

To *truncate* a (d,n)-polytope P at a vertex v, we form the intersection $\tau_v P$ of P with any closed halfspace that misses v and whose bounding hyperplane passes strictly between v and the remaining vertices of P. $\tau_v P$ is a $(d, n+1)$-polytope with a new facet $\tau(v)$. For the details we refer again to [8] or [11]. As in [8], we denote by σX the set of vertices of $\tau(v)$ adjacent to vertices in X.

The following lemma, Lemma 4.2 from [7], establishes when vertex-truncation can be used to produce H-sharp $(d, n+1)$-polytopes from H-sharp (d,n)-polytopes.

Lemma 2.3. *Let $(P, X, Y) \in (d, n\colon h, k)$, $h > 0$, and let v be any vertex of the indicated h-face. Then*

$$(\tau_v P, \sigma X, Y) \in (d, n+1\colon h-1, k).$$

Thus $(d, n+i) \in \mathscr{S}$, $0 \leqslant i \leqslant h+k$.

2.3. Blending

For $i = 1,2$, let x_i be a vertex of a (d, n_i)-polytope P_i. The combinatorial idea behind the blending operation $\bowtie$ is a pairwise identification of the facets of P_1 incident to x_1 with those of P_2 incident to x_2; consequently, each edge incident to x_1 in P_1 is blended with an edge incident to x_2 in P_2 to form a single edge in $P_1 \bowtie P_2$. These blended edges in $P_1 \bowtie P_2$ together with all faces incident to them form the *waist* of $P_1 \bowtie P_2$. The new polytope $P_1 \bowtie P_2$ is of class $(d, n_1 + n_2 - d)$. Geometric constructions of a blend of P_1 and P_2 are recorded in [8,1].

In the context of two triples (P_1, x_1, Y_1) and (P_2, x_2, Y_2), an edge in the waist of $(P_1, x_1) \bowtie (P_2, x_2)$ is either a *fast–slow edge*, a *fast–fast edge*, or a *slow–slow edge*, depending on whether the two edges blended to form it were fast or slow toward the vertices in Y_i. Since we want to construct H-sharp polytopes we consider only blendings for which $\delta(P_1 \bowtie P_2) \geqslant \delta(P_1) + \delta(P_2)$. Such blends do not allow fast–fast edges in the waist of $P_1 \bowtie P_2$ and are thereby called *fast–slow blends*.

We blend H-sharp (d, n_1)-polytopes and H-sharp (d, n_2)-polytopes to form H-sharp $(d, n_1 + n_2 - d)$-polytopes, using the following, Corollary 6.3 from [8].

Lemma 2.4. *If $(d, n_1\colon h_1, k_1), (d, n_2\colon h_2, k_2) \in \mathscr{T}$ and $h_1 + h_2 \geqslant d$, then $(d, n_1 + n_2 - d\colon k_1, k_2) \in \mathscr{T}$.*

There is no barrier to blending when $n_i \leqslant 2d$; in fact, blending a simple polytope with a simplex, $(P, v) \bowtie (T^d, w)$ is combinatorially equivalent to truncating a vertex, $\tau_v P$.

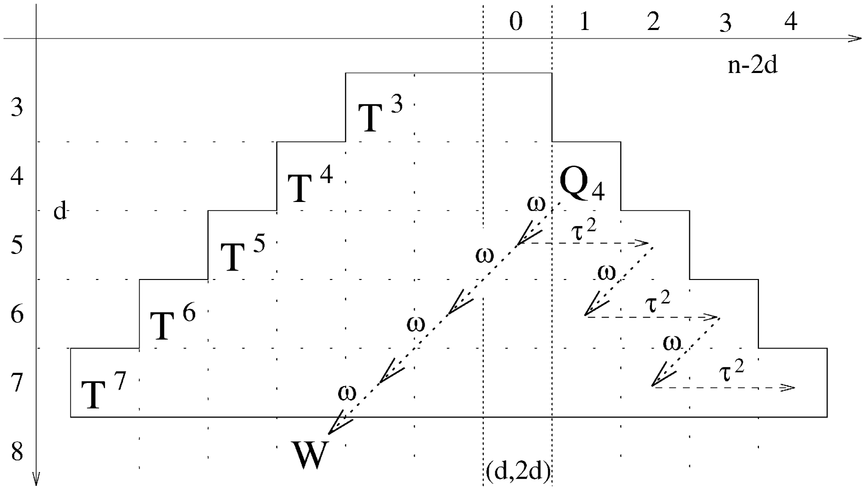

Fig. 1. The tabulation of H-sharp pairs (d,n) for $d \leqslant 7$. Wedging is indicated by ω and truncation by τ. Q_4 is the unique $(4,9)$-polytope of diameter 5, and T^d is the d-simplex.

3. The tabulation for $d < 8$

In Fig. 1 we give a tabulation of all H-sharp pairs (d,n) for $d<8$ which can be obtained by applying the operations of truncation, wedging, and blending to the d-simplex T^d or to Q_4. The operation of product is weaker than the three we consider; from $(d_1,n_1\colon h_1,k_1)$ and $(d_2,n_2\colon h_2,k_2)$, a product produces a triple in $(d_1+d_2,n_1+n_2\colon h_1+h_2,k_1+k_2)$. All known H-sharp simple d-polytopes, with $d<8$, lie within the elements of $\mathscr{T}$ generated by this tabulation.

The d-simplex T^d is shown as the sole occupant of the first diagonal. It demonstrates that for $0 \leqslant j \leqslant d$, $(d,d+1\colon j,d-1-j) \in \mathscr{T}$. We write this succinctly as $(d,d+1\colon [0,d-1]) \subset \mathscr{T}$. In general, $(d,n\colon [h,k])$ denotes the set of $(d,n\colon i,j)$ with $i+j=h+k$ and $\min\{h,k\} \leqslant i,j$. Truncations of the simplex show that $(d,d+k\colon [0,d-k]) \subset \mathscr{T}$ for $1 \leqslant k \leqslant d$.

For $d<8$ the d-simplex and its truncations provide the following elements and subsets of $\mathscr{T}$:

$(3,4\colon [0,2])$, $(3,5\colon 0,1)$, $(3,6\colon 0,0)$,
$(4,5\colon [0,3])$, $(4,6\colon [0,2])$, $(4,7\colon 0,1)$, $(4,8\colon 0,0)$,
$(5,6\colon [0,4])$, $(5,7\colon [0,3])$, $(5,8\colon [0,2])$, $(5,9\colon 0,1)$, $(5,10\colon 0,0)$,
$(6,7\colon [0,5])$, $(6,8\colon [0,4])$, $(6,9\colon [0,3])$, $(6,10\colon [0,2])$, $(6,11\colon 0,1)$, $(6,12\colon 0,0)$,
$(7,8\colon [0,6])$, $(7,9\colon [0,5])$, $(7,10\colon [0,4])$, $(7,11\colon [0,3])$, $(7,12\colon [0,2])$,
$(7,13\colon 0,1)$, $(7,14\colon 0,0)$.

Wedging, blending, or products using only the polytopes included here provide no new elements of $\mathscr{T}$. For example, blending with a simplex or any of its truncations produces at best the same element of $\mathscr{T}$ that successive truncations would; to see this, observe that for each of these quadruples $(d,n\colon h,k)$ produced by the simplex, the relation $2d=n+h+k$ holds, so that from Lemma 2.4 a blend on any $(d,m\colon i,j)$ and one of these $(d,n\colon h,k)$ with $k=d-j$ produces $(d,m+n-d\colon i,j-(n-d))$.

One well-known polytope, Q_4, provides an element of $\mathscr{T}$, $(Q_4,\{x,y\})\in(4,9\colon 0,0)$, which is not provided by the simplex under the operations of truncation of faces, wedging over facets, or fast-slow vertex-blending. Taking wedges and truncations of Q_4 provides the following elements and subsets of $\mathscr{T}$:

$(4,9\colon 0,0)$,
$(5,10\colon 1,1)$, $(5,11\colon 0,1)$, $(5,12\colon 0,0)$,
$(6,11\colon 2,2)$, $(6,12\colon 1,2)$, $(6,13\colon [0,2])$, $(6,14\colon 0,1)$, $(6,15\colon 0,0)$,
$(7,12\colon 3,3)$, $(7,13\colon 2,3)$, $(7,14\colon [1,3])$, $(7,15\colon [0,3])$, $(7,16\colon [0,2])$,
$(7,17\colon 0,1)$, $(7,18\colon 0,0)$.

The operations of wedging, truncation and blending of any polytopes within these elements of $\mathscr{T}$ produce no other quadruples for $d\leqslant 7$. In particular, note that every fast–slow blend possible amongst the polytopes indicated above must be with a simplex or one of its truncations. So under the operations considered here, and given the simplex and Q_4 as representative seeds from the known simple polytopes, the tabulation of $\mathscr{T}$ for $d\leqslant 7$ is complete.

4. All pairs $(8,n\geqslant 9)$ are H-sharp

In this section we use wedging, truncation and blending to show that if $d\geqslant 8$, then the pair (d,n) is H-sharp for all $n>d$.

Lemma 4.1. $(8,13\colon 4,4)\in\mathscr{T}$.

Proof. By Lemma 2.1, $(Q_4,X,Y)\in(4,9\colon 0,0)$ implies $(\omega Q_4,\omega X,\omega Y)\in(5,10\colon 1,1)$. Then wedging three times according to the Lemma 2.2 yields $(W,\tilde{X},\tilde{Y})\in(8,13\colon 4,4)$. $\square$

The incidence matrix for W is of course closely related to that of Q_4. We write the (9×27) incidence matrix of Q_4, $M(Q_4)$, as

$$M(Q_4):=\left(\begin{array}{cccc|cccc|cccc|c|cccc|cccc|cccc|cc}
1&&&&1&&&&1&&&&1&1&&1&&1&&&1&1&1&&1&1&\\
&1&&&&1&&&&1&&&1&&1&&1&1&1&&&1&1&1&&&1\\
&&1&&&&1&&&&1&&1&1&&1&&&1&1&&&1&1&1&1&\\
&&&1&&&&1&&&&1&1&&1&&1&&&1&1&1&&1&1&&1\\
\hline
1&&1&&&1&1&&1&1&&&&1&&&1&1&&&1&1&&&&1&\\
&1&&1&&&1&1&&1&1&&&1&1&&&1&1&&&&1&&&&1\\
1&&1&&1&&&1&&&1&1&&&1&1&&&1&1&&&&1&&1&\\
&1&&1&1&1&&&1&&&1&&&&1&1&&&1&1&&&&1&&1\\
\hline
1&1&1&1&1&1&1&1&1&1&1&1&&&&&&&&&&&&&&&
\end{array}\right)_{(9\times 27)}=\begin{pmatrix} A & B\\ \langle 1\rangle & \langle 0\rangle\end{pmatrix},$$

in which $\langle 1\rangle$ represents a (1×12) 1-vector and $\langle 0\rangle$ a (1×15) 0-vector, indicating whether a vertex is incident to the ninth facet. The (8×12) matrix block A and the (8×15) matrix block B encode the incidences of the vertices to the first eight facets.

In particular the 12 columns of A and $\langle 1\rangle$ represent the incidences for the vertices of the ninth facet, the foot of the first wedge. Under this ordering, the 5-distant vertices are v_{26} and v_{27}, encoded in the last two columns.

After the four wedges prescribed by Lemma 2.2, we have the following (13×87) incidence matrix for W, $M(W)$:

$$M(W)=\begin{pmatrix} A & B & B & B & B & B \\ \langle 1\rangle & \langle 0\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle \\ \langle 1\rangle & \langle 1\rangle & \langle 0\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle \\ \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 0\rangle & \langle 1\rangle & \langle 1\rangle \\ \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 0\rangle & \langle 1\rangle \\ \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 1\rangle & \langle 0\rangle \end{pmatrix}_{13\times 87}.$$

Here the $\langle 0\rangle$ or $\langle 1\rangle$ is a (1×15) or (1×12) vector, of 0's or 1's, respectively, as its placement dictates.

Each of the five images of v_{26} is at distance 5 from each of the five images of v_{27}. The images of v_{26} are the vertices of one simplicial 4-face of W, and the images of v_{27} are the vertices of another simplicial 4-face.

Corollary 4.2. *For all* $d\geqslant 8$ *and all* $n>d$, $(d,n)\in S$.

Proof. It suffices to show that for all $n\geqslant 9$, $(8,n)$ is H-sharp. For this we apply Lemma 2.4 to $(P,X,Y)\in(8,13\colon 4,4)$. Thus $(8,18\colon 4,4)\in\mathscr{T}$; and by iterating the fast-slow blending operation, $(8,8+5k\colon 4,4)\in\mathscr{T}$ for all $k\geqslant 2$. Truncating according to Lemma 2.3 fills in the gaps. $\square$

5. Remarks

In Q_4 there are 16 short paths between the 5-distant pair of vertices, twelve of which are nonrevisiting. Thus, 5-distant vertices in W share a 5-face combinatorially equivalent to ωQ_4, and they are connected by 12 nonrevisiting paths; if these 5-distant vertices lie on the same natural image of Q_4, then they share that 4-face and are connected by four additional revisiting paths.

A fast–slow blend of k copies of W is an $(8,8+5k)$-polytope which we here denote W^k. Although in W no vertices are estranged, in $W^2\in(8,18)$ there are already many pairs of estranged 10-distant vertices. In each W^k there is an H-pair consisting of two 4-simplices; hence there are at least 25 pairs of vertices at distance $5k$.

Through an involved accounting of nonrevisiting paths [6], we can show that in W^k, each of these pairs of $5k$-distant vertices is connected by $\frac{1}{2}\cdot 24^k$ nonrevisiting short paths. Under truncation and wedging these produce $(d,2d)$-polytopes with $\frac{1}{2}\cdot 24^k\cdot 2^j$ d-paths between at least one pair of estranged vertices; here $d=5k+j$. This improves the best-known upper bound for $\#(d,2d)$ to

$$\#(d,2d)\leqslant\tfrac{1}{2}\cdot 24^{\lfloor d/5\rfloor}\cdot 2^{d \bmod 5}.$$

This bound never differs by more than a factor of 1.26 from $\frac{1}{2} \cdot 24^{d/5}$. Rounding $24^{1/5}$ up, we conclude that $\#^d(d, 2d) \in \mathcal{O}(1.88817502259^d)$.

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